

# Lebesgue approximation of $(2, \beta)$ -superprocesses

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**Abstract:** Let  $\xi = (\xi_t)$  be a locally finite  $(2, \beta)$ -superprocess in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ . Then for any fixed  $t > 0$ , the random measure  $\xi_t$  can be a.s. approximated by suitably normalized restrictions of Lebesgue measure to the  $\varepsilon$ -neighborhoods of  $\text{supp } \xi_t$ . This extends the Lebesgue approximation of Dawson-Watanabe superprocesses. Our proof is based on a truncation of  $(\alpha, \beta)$ -superprocesses and uses bounds and asymptotics of hitting probabilities.

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## 1 Introduction

By an  $(\alpha, \beta)$ -superprocess (or  $(\alpha, \beta)$ -process, for short) in  $\mathbb{R}^d$  we mean a vaguely rcll, measure-valued strong Markov process  $\xi = (\xi_t)$  in  $\mathbb{R}^d$  satisfying  $E_\mu e^{-\xi_t f} = e^{-\mu v_t}$  for suitable functions  $f \geq 0$ , where  $v = (v_t)$  is the unique solution to the *evolution equation*  $\dot{v} = \frac{1}{2}\Delta_\alpha v - v^{1+\beta}$  with initial condition  $v_0 = f$ . Here  $\Delta_\alpha = -(-\Delta)^{\alpha/2}$  is the fractional Laplacian,  $\alpha \in (0, 2]$  refers to the spatial motion, and  $\beta \in (0, 1]$  refers to the branching mechanism. When  $\alpha = 2$  and  $\beta = 1$  we get the *Dawson-Watanabe superprocess* (*DW-process* for short), where the spatial motion is standard Brownian motion. General surveys of superprocesses include the excellent monographs and lecture notes [2, 6, 7, 13, 14, 17].

In this paper we consider superprocesses with possibly infinite initial measures. Indeed, by the additivity property of superprocesses, we can construct the  $(\alpha, \beta)$ -process  $\xi$  with any  $\sigma$ -finite initial measure  $\mu$ . In Lemma 2.5 we show that  $\xi_t$  is a.s. locally finite for every  $t > 0$  iff  $\mu p_\alpha(t, \cdot) < \infty$  for all  $t$ , where  $p_\alpha(t, x)$  denotes the transition density of a symmetric  $\alpha$ -stable process. Note that when  $\alpha = 2$ ,  $p_2(t, x) = p_t(x)$  is the standard normal density.

For any measure  $\mu$  on  $\mathbb{R}^d$  and constant  $\varepsilon > 0$ , write  $\mu^\varepsilon$  for the restriction of Lebesgue measure  $\lambda^d$  to the  $\varepsilon$ -neighborhood of  $\text{supp } \mu$ . For a DW-process  $\xi$  in  $\mathbb{R}^d$  with any finite initial measure, Tribe [18] showed that  $\varepsilon^{2-d} \xi_t^\varepsilon \xrightarrow{w} c_d \xi_t$  a.s. as  $\varepsilon \rightarrow 0$  when  $d \geq 3$ , where  $\xrightarrow{w}$  denotes weak convergence and  $c_d > 0$  is a

constant depending on  $d$ . For a locally finite DW-process  $\xi$  in  $\mathbb{R}^2$ , Kallenberg [11] showed that  $\tilde{m}(\varepsilon) |\log \varepsilon| \xi_t^\varepsilon \xrightarrow{v} \xi_t$  a.s. as  $\varepsilon \rightarrow 0$ , where  $\xrightarrow{v}$  denotes vague convergence and  $\tilde{m}$  is a suitable normalizing function. Our main result is Theorem 5.2, where we prove that, for a locally finite  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ ,  $\varepsilon^{2/\beta-d} \xi_t^\varepsilon \xrightarrow{v} c_{\beta,d} \xi_t$  a.s. as  $\varepsilon \rightarrow 0$ , where  $c_{\beta,d} > 0$  is a constant depending on  $\beta$  and  $d$ . In particular, the  $(2, \beta)$ -process  $\xi_t$  distributes its mass over  $\text{supp } \xi_t$  in a deterministic manner, which extends the corresponding property of DW-processes (cf. [7], page 115, or [17], page 212). For DW-processes, this property can also be inferred from some deep results involving the exact Hausdorff measure (cf. [4]). However, for any  $(\alpha, \beta)$ -process  $\xi$  with  $\alpha < 2$ ,  $\text{supp } \xi_t = \mathbb{R}^d$  or  $\emptyset$  a.s. (cf. [8, 16]), and so the corresponding property fails. Our result shows that this property depends only on the spatial motion.

To prove our main result, we adapt the probabilistic approach for DW-processes from [11]. However, the finite variance of DW-processes plays a crucial role there. In order to deal with the infinite variance of  $(2, \beta)$ -processes with  $\beta < 1$ , we use a truncation of  $(\alpha, \beta)$ -processes from [15], which will be further developed in Section 2 of the present paper. By this truncation we may reduce our discussion to the truncated processes, where the variance is finite.

To adapt the probabilistic approach from [11] to study the truncated processes, we also need to develop some technical tools. Thus, in Section 3 we improve the upper bounds of hitting probabilities for  $(2, \beta)$ -processes with  $\beta < 1$  and their truncated processes. As an immediate application, in Theorem 3.3 we improve some known extinction criteria of the  $(2, \beta)$ -process  $\xi$  by showing that the local extinction property  $\xi_t \xrightarrow{d} 0$  and the seemingly stronger support property  $\text{supp } \xi_t \xrightarrow{d} \emptyset$  are equivalent. Then in Section 4 we derive some asymptotic results of these hitting probabilities. In particular, for the  $(2, \beta)$ -process  $\xi$  we show in Theorem 4.3 that  $\varepsilon^{2/\beta-d} P_\mu\{\xi_t B_x^\varepsilon > 0\} \rightarrow c_{\beta,d} (\mu * p_t)(x)$ , where  $B_x^r$  denotes an open ball around  $x$  of radius  $r$ , which extends the corresponding result for DW-processes (cf. Theorem 3.1(b) in [3]). Since the truncated processes don't have the scaling properties of the  $(2, \beta)$ -process, our general method is first to study the  $(2, \beta)$ -process, then to estimate the truncated processes by the  $(2, \beta)$ -process, in order to get the needed results for the truncated processes.

The extension of results of DW-processes to general  $(\alpha, \beta)$ -processes is one of the major themes in the research of superprocesses. Since the spatial motion of the  $(\alpha, \beta)$ -process is not continuous when  $\alpha < 2$  and the  $(\alpha, \beta)$ -process has infinite variance when  $\beta < 1$ , many extensions are not straightforward, and some may not even be valid. However, it turns out that several properties of the support of  $(2, \beta)$ -processes depend only on the spatial motion. These properties include short-time propagation of the support (cf. Theorem 9.3.2.2 in [2]) and Hausdorff dimension of the support (cf. Theorem 9.3.3.5 in [2]). Our result also belongs to that category.

In this paper we are mainly using the notations in [11]. Especially we use relations such as  $\equiv$ ,  $\lesssim$ , and  $\asymp$ , where the first two mean equality and inequality up to a constant factor, and the last one is the combination of  $\lesssim$  and  $\gtrsim$ . Other notation will be explained whenever it occurs.

## 2 Truncated superprocesses and local finiteness

It is well known that the  $(\alpha, 1)$ -process has weakly continuous sample paths. By contrast, the  $(\alpha, \beta)$ -process  $\xi$  with  $\beta < 1$  has only weakly rcll sample paths with jumps of the form  $\Delta\xi_t = r\delta_x$ , for some  $t > 0$ ,  $r > 0$ , and  $x \in \mathbb{R}^d$ . Let

$$N_\xi(dt, dr, dx) = \sum_{(t,r,x): \Delta\xi_t=r\delta_x} \delta_{(t,r,x)}.$$

Clearly the point process  $N_\xi$  on  $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$  records all information about the jumps of  $\xi$ . By the proof of Theorem 6.1.3 in [2], we know that  $N_\xi$  has compensator measure

$$\hat{N}_\xi(dt, dr, dx) = c_\beta(dt)r^{-2-\beta}(dr)\xi_t(dx), \quad (1)$$

where  $c_\beta$  is a constant depending on  $\beta$ . Due to all the “big” jumps,  $\xi_t$  has infinite variance. Some methods for  $(\alpha, 1)$ -processes, which rely on the finite variance of the processes, are not directly applicable to  $(\alpha, \beta)$ -processes with  $\beta < 1$ .

In [15], Mytnik and Villa introduced a truncation method for  $(\alpha, \beta)$ -processes with  $\beta < 1$ , which can be used to study  $(\alpha, \beta)$ -processes with  $\beta < 1$ , especially to extend results of  $(\alpha, 1)$ -processes to  $(\alpha, \beta)$ -processes with  $\beta < 1$ . Specifically, for the  $(\alpha, \beta)$ -process  $\xi$  with  $\beta < 1$ , we define the stopping time  $\tau_K = \inf\{t > 0 : \|\Delta\xi_t\| > K\}$  for any constant  $K > 0$ . Clearly  $\tau_K$  is the time when  $\xi$  has the first jump greater than  $K$ . For any finite initial measure  $\mu$ , they proved that one can define  $\xi$  and a weakly rcll, measure-valued Markov process  $\xi^K$  on a common probability space such that  $\xi_t = \xi_t^K$  for  $t < \tau_K$ . Intuitively,  $\xi^K$  equals  $\xi$  minus all masses produced by jumps greater than  $K$  along with the future evolution of those masses. In this paper, we call  $\xi^K$  the *truncated  $K$ -process* of  $\xi$ . Since all “big” jumps are omitted,  $\xi_t^K$  has finite variance. They also proved that  $\xi_t^K$  and  $\xi_t$  agree asymptotically as  $K \rightarrow \infty$ . We give a different proof of this result, since similar ideas will also be used at several crucial stages later. We write  $P_\mu\{\xi \in \cdot\}$  for the distribution of  $\xi$  with initial measure  $\mu$ .

**Lemma 2.1** *Fix any finite  $\mu$  and  $t > 0$ . Then  $P_\mu\{\tau_K > t\} \rightarrow 1$  as  $K \rightarrow \infty$ .*

*Proof:* If  $\tau_K \leq t$ , then  $\xi$  has at least one jump greater than  $K$  before time  $t$ . Noting that  $N_\xi([0, t], (K, \infty), \mathbb{R}^d)$  is the number of jumps greater than  $K$

before time  $t$ , we get by Theorem 25.22 of [10] and (1),

$$\begin{aligned} P_\mu\{\tau_K \leq t\} &\leq E_\mu N_\xi([0, t], (K, \infty), \mathbb{R}^d) \\ &= E_\mu \hat{N}_\xi([0, t], (K, \infty), \mathbb{R}^d) \\ &\stackrel{(\circ)}{=} K^{-1-\beta} E_\mu \int_0^t \|\xi_s\| ds = t \|\mu\| K^{-1-\beta} \rightarrow 0 \end{aligned}$$

as  $K \rightarrow \infty$ , where the last equation holds by  $E_\mu \|\xi_s\| = \|\mu\|$ .  $\square$

Using Lemma 1 of [15] and a recursive construction, we can prove that  $\xi_t^K(\omega) \leq \xi_t(\omega)$  for any  $t$  and  $\omega$ . So indeed,  $\xi^K$  is a “truncation” of  $\xi$ .

**Lemma 2.2** *We can define  $\xi$  and  $\xi^K$  on a common probability space such that:*

- (i)  $\xi$  is an  $(\alpha, \beta)$ -process with  $\beta < 1$  and a finite initial measure  $\mu$ , and  $\xi^K$  is its truncated  $K$ -process,
- (ii)  $\xi_t(\omega) \geq \xi_t^K(\omega)$  for any  $t$  and  $\omega$ ,
- (iii)  $\xi_t(\omega) = \xi_t^K(\omega)$  for  $t < \tau_K(\omega)$ .

*Proof:* Let  $\xi_{m,n}(t)$  denote the process  $\xi_{m,n}$  at time  $t$ . Use  $D([0, \infty), \hat{\mathcal{M}}_d)$  as our  $\Omega$ , the space of rcll functions from  $[0, \infty)$  to  $\hat{\mathcal{M}}_d$ . We endow  $\Omega$  with the Skorohod  $J_1$ -topology. Let  $\mathcal{A} = \mathcal{B}(\Omega)$ .

Let  $\zeta_1(t, \omega) = \omega(t)$  be an  $(\alpha, \beta)$ -process defined on  $(\Omega, \mathcal{A}, P)$  with initial measure  $\mu$ , and define  $\tau_{K_1} = \inf\{t > 0 : \|\Delta\zeta_1(t)\| > K\}$ . Then define a kernel  $u$  from  $\hat{\mathcal{M}}_d$  to  $\Omega$  such that  $u(\nu, \cdot)$  is the distribution of an  $(\alpha, \beta)$ -process with initial measure  $\nu$ , and a kernel  $u^K$  from  $\hat{\mathcal{M}}_d$  to  $\Omega$  such that  $u^K(\nu, \cdot)$  is the distribution of the truncated  $K$ -process of an  $(\alpha, \beta)$ -process with initial measure  $\nu$ . By Lemma 6.9 in [10], we can define  $\zeta_{1,\infty}$  to be an  $(\alpha, \beta)$ -process with initial measure  $\zeta_1(\tau_{K_1})$  on an extension of  $(\Omega, \mathcal{A}, P)$ , and  $\zeta'_{1,\infty}$  to be the truncated  $K$ -process of an  $(\alpha, \beta)$ -process with initial measure  $\zeta_1(\tau_{K_1}^-)$ . Now define  $\xi_1$  and  $\xi_1^K$  by

$$\begin{aligned} \xi_1(t) &= \begin{cases} \zeta_1(t), & t < \tau_{K_1}, \\ \zeta_{1,\infty}(t - \tau_{K_1}), & t \geq \tau_{K_1}, \end{cases} \\ \xi_1^K(t) &= \begin{cases} \zeta_1(t), & t < \tau_{K_1}, \\ \zeta'_{1,\infty}(t - \tau_{K_1}), & t \geq \tau_{K_1}. \end{cases} \end{aligned}$$

By the strong Markov property of  $(\alpha, \beta)$ -processes and the above construction, we can verify that  $\xi_1$  is an  $(\alpha, \beta)$ -process. By Lemma 1 in [15],  $\xi_1^K$  is the truncated  $K$ -process of an  $(\alpha, \beta)$ -process. Moreover,  $\xi_1$  and  $\xi_1^K$  satisfy conditions (ii) and (iii) on  $[0, \tau_{K_1})$ .

Let  $u'$  be a kernel from  $\hat{\mathcal{M}}_d \times \hat{\mathcal{M}}_d$  to  $\mathcal{A} \times \mathcal{A}$  such that  $u'(\nu, \nu', \cdot, \cdot)$  is the distribution of a pair of two independent  $(\alpha, \beta)$ -processes with initial measures  $\nu$  and  $\nu'$  respectively. Define  $(\zeta_{2,0}, \zeta_{2,1})$  with distribution

$$u'(\xi_1^K(\tau_{K_1}^-), \xi_1(\tau_{K_1}) - \xi_1^K(\tau_{K_1}^-), \cdot, \cdot).$$

Let  $\zeta_2 = \zeta_{2,0} + \zeta_{2,1}$ ,  $\zeta'_2 = \zeta_{2,0}$ , and  $\tau_{K_2} = \inf\{t > 0 : \|\Delta\zeta_2(t)\| > K\}$ . Let  $\zeta_{2,\infty}$  be an  $(\alpha, \beta)$ -process with initial measure  $\zeta_2(\tau_{K_2})$ , and let  $\zeta'_{2,\infty}$  be the truncated  $K$ -process of an  $(\alpha, \beta)$ -process with initial measure  $\zeta'_2(\tau_{K_2}^-)$ . Now define  $\xi_2$  and  $\xi_2^K$  by

$$\xi_2(t) = \begin{cases} \xi_1(t), & t < \tau_{K_1}, \\ \zeta_2(t - \tau_{K_1}), & \tau_{K_1} \leq t < \tau_{K_1} + \tau_{K_2}, \\ \zeta_{2,\infty}(t - \tau_{K_1} - \tau_{K_2}), & t \geq \tau_{K_1} + \tau_{K_2}, \end{cases}$$

$$\xi_2^K(t) = \begin{cases} \xi_1^K(t), & t < \tau_{K_1}, \\ \zeta'_2(t - \tau_{K_1}), & \tau_{K_1} \leq t < \tau_{K_1} + \tau_{K_2}, \\ \zeta'_{2,\infty}(t - \tau_{K_1} - \tau_{K_2}), & t \geq \tau_{K_1} + \tau_{K_2}. \end{cases}$$

Similarly,  $\xi_2$  is an  $(\alpha, \beta)$ -process and  $\xi_2^K$  is the truncated  $K$ -process of an  $(\alpha, \beta)$ -process. They satisfy conditions (ii) and (iii) on  $[0, \tau_{K_1} + \tau_{K_2}]$ .

Continue the above construction: For every  $n$ , define  $\xi_n$  and  $\xi_n^K$  such that  $\xi_n$  is an  $(\alpha, \beta)$ -process,  $\xi_n^K$  is the truncated  $K$ -process of an  $(\alpha, \beta)$ -process, and they satisfy conditions (ii) and (iii) on  $[0, \sum_{k=1}^n \tau_{K_k}]$ .

It suffices to prove that  $\sum_{k=1}^{\infty} \tau_{K_k} = \infty$  a.s. Suppose  $P(\sum_{k=1}^{\infty} \tau_{K_k} < \infty) > 0$ . Then there exist  $t$  and  $a$  such that  $P(\sum_{k=1}^{\infty} \tau_{K_k} < t) = a > 0$ . Since for every  $n$ ,  $\xi_n$  is an  $(\alpha, \beta)$ -process with initial measure  $\mu$ , we get

$$an \leq E_{\mu} \hat{N}_{\xi_n}([0, t], (K, \infty), \mathbb{R}^d).$$

Noting that by (1)  $E_{\mu} \hat{N}_{\xi_n}([0, t], (K, \infty), \mathbb{R}^d)$  is the same finite constant for different  $n$ , we get a contradiction. So  $\sum_{k=1}^{\infty} \tau_{K_k} = \infty$  a.s.  $\square$

Just as the DW-process, the  $(\alpha, \beta)$ -process  $\xi$  and its truncated  $K$ -process  $\xi^K$  also have cluster structures (cf. Section 3 in [4]). Specifically, for any fixed  $t$ ,  $\xi_t$  is a Cox cluster process, such that the “ancestors” of  $\xi_t$  at time  $s = t - h$  form a Cox process directed by  $(\beta h)^{-1/\beta} \xi_s$ , and the generated  $h$ -clusters  $\eta_h^i$  are conditionally independent and identically distributed apart from shifts. For the truncated  $K$ -process  $\xi^K$ , the situation is similar, except that the clusters are different (because of the truncation) and the term  $(\beta h)^{-1/\beta}$  for  $\xi$  needs to be replaced by  $a_K(h)$  (or  $a_h$ , when  $K$  is fixed). Use  $\eta_h^{K,i}$  (or  $\eta_h^{K,i}$ ) to denote the generated  $h$ -clusters of  $\xi^K$ . Write  $P_x\{\eta_t \in \cdot\}$  for the distribution of  $\eta_t$  centered at  $x \in \mathbb{R}^d$ , and define  $P_{\mu}\{\eta_t \in \cdot\} = \int \mu(dx) P_x\{\eta_t \in \cdot\}$ . The following comparison of  $a_K(h)$  and  $(\beta h)^{-1/\beta}$ , although not used explicitly in the present paper, should be useful in other applications of the truncation method.

**Lemma 2.3** *Fix any  $K > 0$ . Then as  $h \rightarrow 0$ ,*

$$(\beta h)^{1/\beta} \leq a_K(h) \leq 2(\beta h)^{1/\beta}.$$

*Proof:* From Lemma 3.4 of [4] we know that

$$(\beta h)^{1/\beta} = \lim_{\theta \rightarrow \infty} 1/v_0(h, \theta),$$

where  $v_0(h, \theta)$  is the solution of  $\dot{v} = -v^{1+\beta}$  with initial condition  $v \equiv \theta$ , and

$$A_K(h) = \lim_{\theta \rightarrow \infty} 1/v_1(h, \theta),$$

where  $v_1(h, \theta)$  is the solution of (1.12) in [15] with initial condition  $v \equiv \theta$ . Define  $M_K(\lambda) = C_\beta(K)\lambda + \Phi^K(\lambda)$ , where  $C_\beta(K)$  and  $\Phi^K$  are such as in (1.12) of [15]. Then  $M_K$  satisfies

$$\lambda^{1+\beta} \leq M_K(\lambda) \text{ and } \lim_{\lambda \rightarrow \infty} \frac{M_K(\lambda)}{\lambda^{1+\beta}} = 1.$$

Clearly it is enough to show that  $(1/2)v_0(h, \theta) \leq v_1(h, \theta) \leq v_0(h, \theta)$  as  $h \rightarrow 0$  and  $\theta \rightarrow \infty$ . This follows from the above properties of  $M_K$ .  $\square$

Unlike the normal densities, we have no explicit expressions for the transition densities of symmetric  $\alpha$ -stable processes when  $\alpha < 2$ . However, a simple estimate of  $p_\alpha(t, x)$  is enough for our needs.

**Lemma 2.4** *Let  $p_\alpha(t, x)$ ,  $\alpha \in (0, 2]$ ,  $t > 0$ , and  $x \in \mathbb{R}^d$ , denote the transition densities of a symmetric  $\alpha$ -stable process on  $\mathbb{R}^d$ . Then for any fixed  $\alpha$  and  $d$ ,*

$$p_\alpha(t, x + y) \leq p_\alpha(2t, x), \quad |y|^\alpha \leq t.$$

*Proof:* First let  $\alpha = 2$ . Note that  $p_2(t, x) = p_t(x)$  is the standard normal density on  $\mathbb{R}^d$ . For  $|x| \leq 4\sqrt{t}$ , trivially  $p_t(x + y) \leq p_{2t}(x)$ . For  $|x| > 4\sqrt{t}$ , it suffices to check that

$$-\frac{|x + y|^2}{2t} \leq -\frac{|x|^2}{4t},$$

that is,  $2|x + y|^2 \geq |x|^2$ , which follows easily from  $|x| \geq 4|y|$ .

Now let  $\alpha < 2$ . By the arguments after Remark 5.3 of [1],

$$p_\alpha(t, x) \asymp \left( t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right). \quad (2)$$

Choose  $K > 2^{1/\alpha}$  to satisfy  $1 \leq 2(1 - 1/K)^{d+\alpha}$ . Since  $|y| \leq t^{1/\alpha}$ , we have for  $|x| > Kt^{1/\alpha}$ ,

$$\frac{t}{|x + y|^{d+\alpha}} \leq \frac{2t}{|x|^{d+\alpha}}.$$

Noticing also that  $(2t)/|x|^{d+\alpha} < (2t)^{-d/\alpha}$  for  $|x| > Kt^{1/\alpha}$ , we get  $p_\alpha(t, x + y) \leq p_\alpha(2t, x)$  for  $|y| \leq t^{1/\alpha}$  and  $|x| > Kt^{1/\alpha}$ . The same inequality holds trivially for  $|y| \leq t^{1/\alpha}$  and  $|x| \leq Kt^{1/\alpha}$ .  $\square$

Using Lemma 2.2 and Lemma 2.4, we can generalize Lemma 3.2 in [11] to any  $(\alpha, \beta)$ -process.

**Lemma 2.5** *Let  $\xi$  be an  $(\alpha, \beta)$ -process in  $\mathbb{R}^d$ ,  $\alpha \in (0, 2]$  and  $\beta \in (0, 1]$ , and fix any  $\sigma$ -finite measure  $\mu$ . Then for any fixed  $t > 0$ , the following two conditions are equivalent:*

- (i)  $\xi_t$  is locally finite a.s.  $P_\mu$ ,
- (ii)  $E_\mu \xi_t$  is locally finite.

Furthermore, (i) and (ii) hold for every  $t > 0$  iff

- (iii)  $\mu p_t < \infty$  for all  $t > 0$ ,

and if  $\alpha < 2$ , then (iii) is equivalent to

- (iv)  $\mu p_t < \infty$  for some  $t > 0$ .

*Proof:* The formulas for  $E_\mu \xi_t$  and  $E_\mu \xi_t^2$  (when  $\beta < 1$ ), well known for finite  $\mu$ , as well as the formulas in Lemma 3 of [15], extend by monotone convergence to any  $\sigma$ -finite measure  $\mu$ . We also need the simple inequality that for any fixed  $\alpha < 2$ ,  $s$ , and  $t$ ,

$$p_\alpha(s, x) \asymp p_\alpha(t, x). \quad (3)$$

To prove it, use (2) and consider three cases:  $|x| \leq (s \wedge t)^{1/\alpha}$ ,  $|x| \geq (s \vee t)^{1/\alpha}$ , and  $(s \wedge t)^{1/\alpha} < |x| < (s \vee t)^{1/\alpha}$ .

If  $\alpha = 2$  and  $\beta = 1$ , then this is Lemma 3.2 of [11]. For  $\alpha < 2$  and  $\beta = 1$ , using Lemma 2.4 and (3) we can proceed as in Lemma 3.2 of [11]. For example, for any fixed  $t > 0$  and  $x \in \mathbb{R}^d$ ,  $p(t, x - u) \lesssim p(|x|^{1/\alpha}, x - u) \lesssim p(2|x|^{1/\alpha}, -u) = p(2|x|^{1/\alpha}, u)$  yields  $\mu * p(t, \cdot)(x) < \infty$ .

Now assume  $\beta < 1$ . Condition (ii) clearly implies (i). Conversely, suppose that  $E_\mu \xi_t B = \infty$  for some  $B$ . Then  $E_\mu \xi_t^K B = \infty$  for any fixed  $K > 0$  by Lemma 3 of [15]. Also, we get by Lemma 3 of [15],

$$P_\mu \left\{ \frac{\xi_t^K B}{E_\mu \xi_t^K B} > r \right\} \geq (1 - r)^2 \frac{(E_\mu \xi_t^K B)^2}{E_\mu (\xi_t^K B)^2} \geq \frac{(1 - r)^2}{1 + ct (E_\mu \xi_t^K B)^{-1}}$$

for any  $r \in (0, 1)$ . Arguing as in the proof of Lemma 3.2 in [11], we get  $\xi_t^K B = \infty$  a.s., and so  $\xi_t B = \infty$  a.s. by Lemma 2.2. In particular, this shows that (i) implies (ii). To prove the equivalence of (ii) and (iii), again using Lemma 2.4 and (3) we can proceed as in Lemma 3.2 of [11]. The last assertion is obvious from (3).  $\square$

### 3 Hitting bounds and neighborhood measures

The Lebesgue approximation depends crucially on estimates of the hitting probability  $P_\mu \{\xi_t B_0^\varepsilon > 0\}$ . In this section, we first estimate  $P_\mu \{\xi_t B_0^\varepsilon > 0\}$  and  $P_\mu \{\xi_t^K B_0^\varepsilon > 0\}$ . Then we use these estimates to study multiple hitting and neighborhood measures of the clusters  $\eta_h^K$  associated with the truncated  $K$ -process  $\xi^K$ . We begin with a well-known relationship between the hitting probabilities of  $\xi_t$  and  $\eta_t$ , which can be proved as in Lemma 4.1 of [11].

**Lemma 3.1** *Let the  $(\alpha, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with associated clusters  $\eta_t$  be locally finite under  $P_\mu$ , let  $\xi^K$  be its truncated  $K$ -process with associated clusters  $\eta_t^K$ , and fix any  $B \in \mathcal{B}^d$ . Then*

$$\begin{aligned} P_\mu\{\eta_t B > 0\} &= -(\beta t)^{1/\beta} \log(1 - P_\mu\{\xi_t B > 0\}), \\ P_\mu\{\xi_t B > 0\} &= 1 - \exp\left(-(\beta t)^{-1/\beta} P_\mu\{\eta_t B > 0\}\right), \\ P_\mu\{\eta_t^K B > 0\} &= -a_t \log(1 - P_\mu\{\xi_t^K B > 0\}), \\ P_\mu\{\xi_t^K B > 0\} &= 1 - \exp(-a_t^{-1} P_\mu\{\eta_t^K B > 0\}). \end{aligned}$$

*In particular,  $P_\mu\{\xi_t B > 0\} \sim (\beta t)^{-1/\beta} P_\mu\{\eta_t B > 0\}$  and  $P_\mu\{\xi_t^K B > 0\} \sim a_t^{-1} P_\mu\{\eta_t^K B > 0\}$  as either side tends to 0.*

Upper and lower bounds of  $P_\mu\{\xi_t B_0^\varepsilon > 0\}$  have been obtained by Delmas [5], using the Brownian snake. However, in this paper we need the following improved upper bound.

**Lemma 3.2** *Let  $\eta_t$  be the clusters of a  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ , let  $\eta_t^K$  be the clusters of  $\xi^K$ , the truncated  $K$ -process of  $\xi$ , and consider a  $\sigma$ -finite measure  $\mu$  on  $\mathbb{R}^d$ . Then for  $0 < \varepsilon \leq \sqrt{t}$ ,*

- (i)  $\mu p_{t'} \leq \varepsilon^{2/\beta-d} (\beta t)^{-1/\beta} P_\mu\{\eta_t B_0^\varepsilon > 0\} \leq \mu p_{2t}$ , where  $t' = \beta t / (1 + \beta)$ ,
- (ii)  $\varepsilon^{2/\beta-d} a_t^{-1} P_\mu\{\eta_t^K B_0^\varepsilon > 0\} \leq \mu p_{2t}$ .

*Proof:* (i) From the proof of Theorem 2.3 in [5] we know that

$$P_x\{\xi_t B_0^\varepsilon > 0\} = 1 - \exp(-N_x\{Y_t B_0^\varepsilon > 0\}),$$

where  $N_x$  and  $Y_t$  are defined in Section 4.2 of [5]. Comparing this with Lemma 3.1 yields

$$(\beta t)^{-1/\beta} P_x\{\eta_t B_0^\varepsilon > 0\} = N_x\{Y_t B_0^\varepsilon > 0\}.$$

By Proposition 6.2 in [5] we get the lower bound. For our upper bound, we will now improve the upper bound in Proposition 6.1 of [5].

For  $0 < \varepsilon/2 < \sqrt{t}$ , define

$$\begin{aligned} \Delta &= \{(r, y) \in \mathbb{R}^+ \times \mathbb{R}^d, r < t, |y| > \varepsilon/2\} \\ &\quad \cup \{(r, y) \in \mathbb{R}^+ \times \mathbb{R}^d, r < t - \varepsilon^2/16, |y| \leq \varepsilon/2\}. \end{aligned}$$

Following the proof of Proposition 6.1 in [5], we have

$$(\beta t)^{-1/\beta} P_x\{\eta_t B_0^\varepsilon > 0\} \leq \varepsilon^{-2/\beta} P_0\{\gamma_s \in B_x^{\varepsilon/2} \text{ for some } s \in [t - \varepsilon^2/16, t)\},$$

where  $\gamma$  is a standard Brownian motion. Define

$$T = \inf\{s \geq t - \varepsilon^2/16 : \gamma_s \in B_x^{\varepsilon/2}\}.$$



Then  $\{T < t\} = \{\gamma_s \in B_x^{\varepsilon/2} \text{ for some } s \in [t - \varepsilon^2/16, t)\}$ . To get our upper bound, it remains to show that

$$P_0\{T < t\} \leq \varepsilon^d p_{2t}(x).$$

To prove this, we need the elementary fact that for any  $x \in \mathbb{R}^d$ ,  $\varepsilon > 0$ ,  $y \in B_x^{\varepsilon/2}$ , and  $s \leq s' = \varepsilon^2/16$ ,

$$P_y\{\gamma_s \notin B_x^\varepsilon\} \leq P_z\{\gamma_{s'} \notin B_x^\varepsilon\} \leq P_z\{\gamma_{s'} \in B_x^\varepsilon\} \leq P_y\{\gamma_s \in B_x^\varepsilon\},$$

where  $z$  is a point on the surface of  $B_x^{\varepsilon/2}$ , and the third relation holds since  $P_z\{\gamma_{s'} \notin B_x^\varepsilon\}$  and  $P_z\{\gamma_{s'} \in B_x^\varepsilon\}$  are both positive constants. Now return to  $P_0\{T < t\}$ . Noting  $t - T \leq \varepsilon^2/16$  on  $\{T < t\}$ , we get

$$\begin{aligned} P_0\{T < t\} &= P_0\{T < t, \gamma_t \in B_x^\varepsilon\} + P_0\{T < t, \gamma_t \notin B_x^\varepsilon\} \\ &= P_0\{T < t, \gamma_t \in B_x^\varepsilon\} + P_0\{P_{\gamma_T}\{\gamma_{t-T} \notin B_x^\varepsilon\}, T < t\} \\ &\leq P_0\{T < t, \gamma_t \in B_x^\varepsilon\} + P_0\{P_{\gamma_T}\{\gamma_{t-T} \in B_x^\varepsilon\}, T < t\} \\ &= P_0\{T < t, \gamma_t \in B_x^\varepsilon\} + P_0\{T < t, \gamma_t \in B_x^\varepsilon\} \\ &\leq P_0\{\gamma_t \in B_x^\varepsilon\} \leq \varepsilon^d p_{2t}(x), \end{aligned}$$

where the second and fourth relations hold by the strong Markov property of Brownian motion and the last relation holds by Lemma 2.4.

(ii) This is obvious from (i), Lemma 2.2, and Lemma 3.1.  $\square$

As an immediate application of the improved upper bound, we may improve some known extinction criteria for  $(2, \beta)$ -processes in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ . This extends Theorem 4.5 of [11] for DW-processes of dimension  $d \geq 2$ . Note that the properties  $\xi_t \xrightarrow{d} 0$  and  $\text{supp } \xi_t \xrightarrow{d} \emptyset$  are given by  $\xi_t B \xrightarrow{P} 0$  and  $1\{\xi_t B > 0\} \xrightarrow{P} 0$ , respectively, for any bounded Borel set  $B$ .

**Theorem 3.3** *Let  $\xi$  be a locally finite  $(2, \beta)$ -process in  $\mathbb{R}^d$ ,  $\beta < 1$  and  $d > 2/\beta$ , with arbitrary initial distribution. Then these conditions are equivalent as  $t \rightarrow \infty$ :*

- (i)  $\xi_t \xrightarrow{d} 0$ ,
- (ii)  $\text{supp } \xi_t \xrightarrow{d} \emptyset$ ,
- (iii)  $\xi_0 p_t \xrightarrow{P} 0$ .

*Proof:* By Lemma 3.1 and Lemma 3.2(i) we get for any fixed  $r$

$$P_\mu\{\xi_t B_0^r > 0\} \leq (\beta t)^{-1/\beta} P_\mu\{\eta_t B_0^r > 0\} \leq \mu p_{2t},$$

and so  $P_\mu\{\xi_t B_0^r > 0\} \leq \mu p_{2t} \wedge 1$ . For a general initial distribution,

$$P\{\xi_t B_0^r > 0\} \leq E(\xi_0 p_{2t} \wedge 1),$$

which shows that (iii) implies (ii). Since clearly (ii) implies (i), it remains to prove that (i) implies (iii).

Let  $\xi$  be locally finite under  $P_\mu$ . We first choose  $f \in C_c^{++}(\mathbb{R}^d)$  with  $\text{supp} f \in B_0^1$ , where  $C_c^{++}(\mathbb{R}^d)$  is such as in Proposition 2.6 of [12]. Clearly  $\xi_t f \xrightarrow{P} 0$  if  $\xi_t B_0^1 \xrightarrow{P} 0$ . By dominated convergence

$$\exp(-\mu v_t) = E_\mu \exp(-\xi_t f) \rightarrow 1,$$

and so  $\mu v_t \rightarrow 0$ . By Proposition 2.6 of [12], we have for  $t$  large enough

$$p_{t/2}(x) \asymp \phi(t/2, x) \leq v_t(x),$$

and so  $\mu p_{t/2} \rightarrow 0$ . For general  $\xi_0$ , we may proceed as in the proof of Theorem 4.5 in [11].  $\square$

The following simple fact is often useful to extend results for finite initial measures  $\mu$  to the general case.

**Lemma 3.4** *Let the  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$  be locally finite under  $P_\mu$ , and suppose that  $\mu \geq \mu_n \downarrow 0$ . Then  $P_{\mu_n}\{\xi_t B > 0\} \rightarrow 0$  as  $n \rightarrow \infty$  for any fixed  $t > 0$  and  $B \in \mathcal{B}^d$ .*

*Proof:* Follow the proof of Lemma 4.3 in [11], then use Lemma 2.5, Lemma 3.1, and Lemma 3.2(i).  $\square$

As in [11] we need to estimate the probability that a ball in  $\mathbb{R}^d$  is hit by more than one subcluster of the truncated  $K$ -process  $\xi^K$ . This is where the truncation of  $\xi$  is needed.

**Lemma 3.5** *Fix any  $K > 0$ . Let  $\xi^K$  be the truncated  $K$ -process of a  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ . For any  $t \geq h > 0$  and  $\varepsilon > 0$ , let  $\kappa_h^\varepsilon$  be the number of  $h$ -clusters of  $\xi_t^K$  hitting  $B_0^\varepsilon$  at time  $t$ . Then for  $\varepsilon^2 \leq h \leq t$ ,*

$$E_\mu \kappa_h^\varepsilon (\kappa_h^\varepsilon - 1) \leq \varepsilon^{2(d-2/\beta)} \left( h^{1-d/2} \mu p_t + (\mu p_{2t})^2 \right).$$

*Proof:* Follow Lemma 4.4 in [11], then use Lemma 3 of [15] and Lemma 3.2(ii).  $\square$

Now we consider the neighborhood measures of the clusters  $\eta_h^K$  associated with the truncated  $K$ -process  $\xi^K$ . For any measure  $\mu$  on  $\mathbb{R}^d$  and constant  $\varepsilon > 0$ , we define the associated *neighborhood measure*  $\mu^\varepsilon$  as the restriction of Lebesgue measure  $\lambda^d$  to the  $\varepsilon$ -neighborhood of  $\text{supp } \mu$ , so that  $\mu^\varepsilon$  has Lebesgue density  $1\{\mu B_x^\varepsilon > 0\}$ . Let  $p_h^{K,\varepsilon}(x) = P_x\{\eta_h^K B_0^\varepsilon > 0\}$ , where the  $\eta_h^K$  are clusters of  $\xi^K$ . Write  $p_h^{K,\varepsilon}(x) = p_h^{K\varepsilon}(x)$  and  $(\eta_h^{K,i})^\varepsilon = \eta_h^{K i\varepsilon}$  for convenience.

**Lemma 3.6** *Let  $\xi^K$  be the truncated  $K$ -process of a  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ . Let the  $\eta_h^{Ki}$  be conditionally independent  $h$ -clusters of  $\xi^K$ , rooted at the points of a Poisson process  $\zeta$  with  $E\zeta = \mu$ . Fix any measurable function  $f \geq 0$  on  $\mathbb{R}^d$ . Then,*

- (i)  $E_\mu \sum_i \eta_h^{Ki\varepsilon} = (\mu * p_h^{K\varepsilon}) \cdot \lambda^d$ ,
- (ii)  $E_\mu \text{Var} \left[ \sum_i \eta_h^{Ki\varepsilon} f | \zeta \right] \leq a_h \varepsilon^{d-2/\beta} h^{d/2} \|f\|^2 \|\mu\|$  for  $\varepsilon^2 \leq h$ .

*Proof:* (i) Follow the proof of Lemma 6.2 (i) in [11].

(ii) First,

$$\text{Var}_x(\eta_h^{K\varepsilon} f) \leq E_x(\eta_h^{K\varepsilon} f)^2 \leq E_x \|\eta_h^{K\varepsilon}\|^2 \|f\|^2 = \|f\|^2 E_x \|\eta_h^{K\varepsilon}\|^2.$$

For  $E_x \|\eta_h^{K\varepsilon}\|^2$ , using Cauchy inequality and Lemma 3.2(ii), we get

$$\begin{aligned} E_x \|\eta_h^{K\varepsilon}\|^2 &= E_x \left( \int 1\{\eta_h^K B_y^\varepsilon > 0\} dy \int 1\{\eta_h^K B_z^\varepsilon > 0\} dz \right) \\ &= \int \int P_x \left( \{\eta_h^K B_y^\varepsilon > 0\} \cap \{\eta_h^K B_z^\varepsilon > 0\} \right) dy dz \\ &\leq \int \int (P_x \{\eta_h^K B_y^\varepsilon > 0\} P_x \{\eta_h^K B_z^\varepsilon > 0\})^{1/2} dy dz \\ &\leq a_h \varepsilon^{d-2/\beta} \int \int (p_{2h}(y-x) p_{2h}(z-x))^{1/2} dy dz \\ &\stackrel{(\ast)}{=} a_h \varepsilon^{d-2/\beta} h^{d/2} \int \int p_{4h}(y-x) p_{4h}(z-x) dy dz \\ &= a_h \varepsilon^{d-2/\beta} h^{d/2}. \end{aligned}$$

Hence, by independence

$$E_\mu \text{Var} \left[ \sum_i \eta_h^{Ki\varepsilon} f | \zeta \right] = E_\mu \int \zeta(dx) \text{Var}_x(\eta_h^{K\varepsilon} f) \leq a_h \varepsilon^{d-2/\beta} h^{d/2} \|f\|^2 \|\mu\|.$$

□

We also need to estimate the overlap between subclusters.

**Lemma 3.7** *Let  $\xi^K$  be the truncated  $K$ -process of a  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ . For any fixed  $t > 0$ , let  $\eta_h^{Ki}$  denote the subclusters in  $\xi^K$  of age  $h > 0$ . Fix any  $\mu \in \hat{\mathcal{M}}_d$ . Then as  $\varepsilon^2 \leq h \rightarrow 0$ ,*

$$E_\mu \left\| \sum_i \eta_h^{Ki\varepsilon} - \xi_t^{K\varepsilon} \right\| \leq \varepsilon^{2(d-2/\beta)} h^{1-d/2}.$$

*Proof:* Follow the proof of Lemma 6.3(i) in [11], then use Lemma 3.2(ii). □

## 4 Hitting asymptotics

For a DW-process  $\xi$  of dimension  $d \geq 3$ , we know from Theorem 3.1(b) of Dawson, Iscoe, and Perkins [3] that, as  $\varepsilon \rightarrow 0$ ,

$$\varepsilon^{2-d} P_\mu \{ \xi_t B_x^\varepsilon > 0 \} \rightarrow c_d (\mu * p_t)(x),$$

uniformly for bounded  $\|\mu\|$ , bounded  $t^{-1}$ , and  $x \in \mathbb{R}^d$ . A similar result for DW-processes of dimension  $d = 2$  is Theorem 5.3(ii) of [11]. In this section, using Lemma 3.2(i), we can prove the corresponding result for  $(2, \beta)$ -processes in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ .

First we fix a continuous function  $f$  on  $\mathbb{R}^d$  such that  $0 < f(x) \leq 1$  for  $x \in B_0^1$  and  $f(x) = 0$  otherwise. Let  $v_\lambda$  be the solution of  $\dot{v} = \frac{1}{2}\Delta v - v^{1+\beta}$  with initial condition  $v(0) = \lambda f$ . Since  $v_\lambda$  is increasing in  $\lambda$ , we can define  $v_\infty = \lim_{\lambda \rightarrow \infty} v_\lambda$ . Using Lemma 3.2(i), we can get an upper bound of  $v_\infty$ , similar to Lemma 3.2 in [3].

**Lemma 4.1** *For any  $t \geq 1$  and  $x \in \mathbb{R}^d$ ,  $v_\infty(t, x) \leq p(2t, x)$ .*

*Proof:* Letting  $\lambda \rightarrow \infty$  in  $E_x \exp(-\xi_t \lambda f) = \exp[-v_\lambda(t, x)]$ , we get

$$P_x \{ \xi_t B_0^1 > 0 \} = 1 - \exp[-v_\infty(t, x)].$$

Comparing this with Lemma 3.1 yields

$$v_\infty(t, x) = (\beta t)^{-1/\beta} P_x \{ \eta_t B_0^1 > 0 \}. \quad (4)$$

Now Lemma 4.1 follows from Lemma 3.2(i).  $\square$

As in Lemma 3.3 of [3], we can apply a PDE result to get the uniform convergence of  $v_\infty$ . Notice that the improved upper bound in Lemma 3.2(i) is crucial here.

**Lemma 4.2** *There exists a constant  $c_{\beta,d} > 0$  such that*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-d} v_\infty(\varepsilon^{-2}t, \varepsilon^{-1}x) = c_{\beta,d} \cdot p(t, x).$$

*The convergence is uniform for bounded  $t^{-1}$  and  $x \in \mathbb{R}^d$ .*

*Proof:* We follow the proof of Lemma 3.3 in [3]. By Lemma 4.1,  $v_\infty(t, x)$  is finite for any  $t \geq 1$  and  $x \in \mathbb{R}^d$ . Then by a standard regularity argument in PDE theory,

$$\dot{v}_\infty = \frac{1}{2}\Delta v_\infty - v_\infty^{1+\beta} \quad (5)$$

on  $[1, \infty) \times \mathbb{R}^d$ . By Lemma 4.1,  $v_\infty(1) \in L^1(\mathbb{R}^d)$ . Set

$$w_\varepsilon(t, x) = \varepsilon^{-d} u_\infty(1 + \varepsilon^{-2}t, \varepsilon^{-1}x).$$

Then by (5),  $\dot{w}_\varepsilon = \frac{1}{2}\Delta w_\varepsilon - \varepsilon^{d-2}w_\varepsilon^{1+\beta}$  with initial condition  $w_\varepsilon(0, x) = \varepsilon^{-d}u_\infty(1, \varepsilon^{-1}x)$ .

Applying Proposition 3.1 in [9] gives

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-d}v_\infty(1 + \varepsilon^{-2}t, \varepsilon^{-1}x) = c_{\beta,d} \cdot p(t, x),$$

uniformly on compact subsets of  $(0, \infty) \times \mathbb{R}^d$ . Together with Lemma 4.1 this yields the uniform convergence on  $[a, \infty) \times \mathbb{R}^d$  for any  $a > 0$ . Moreover, letting  $t = t' - \varepsilon^2$ , we get

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-d}v_\infty(\varepsilon^{-2}t', \varepsilon^{-1}x) = c_{\beta,d} \cdot p(t', x),$$

uniformly on  $[a, \infty) \times \mathbb{R}^d$  for any  $a > 0$ .

It remains to prove that  $c_{\beta,d} > 0$ . Using (4) and the lower bound in Lemma 3.2(i), we obtain

$$\begin{aligned} \varepsilon^{-d}v_\infty(\varepsilon^{-2}t, \varepsilon^{-1}x) &= \varepsilon^{-d}(\beta t)^{-1/\beta} P_{\varepsilon^{-1}x} \{ \eta_{\varepsilon^{-2}t} B_0^1 > 0 \} \\ &\geq \varepsilon^{-d} p \left( \frac{\beta \varepsilon^{-2}t}{1 + \beta}, \varepsilon^{-1}x \right) = p \left( \frac{\beta t}{1 + \beta}, x \right), \end{aligned}$$

and so  $c_{\beta,d} > 0$ . □

Now we can derive the asymptotic hitting rate for a  $(2, \beta)$ -process.

**Theorem 4.3** *Let the  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$  be locally finite under  $P_\mu$ . Fix any  $t > 0$  and  $x \in \mathbb{R}^d$ . Then as  $\varepsilon \rightarrow 0$ ,*

$$\varepsilon^{2/\beta-d} P_\mu \{ \xi_t B_x^\varepsilon > 0 \} \rightarrow c_{\beta,d}(\mu * p_t)(x).$$

*The convergence is uniform for bounded  $\|\mu\|$ , bounded  $t^{-1}$ , and  $x \in \mathbb{R}^d$ . Similar results hold for the clusters  $\eta_t$  with  $p_t$  replaced by  $(\beta t)^{1/\beta} p_t$ .*

*Proof:* We first prove that as  $\varepsilon \rightarrow 0$ ,

$$\varepsilon^{2/\beta-d}(\beta t)^{-1/\beta} P_\mu \{ \eta_t B_x^\varepsilon > 0 \} \rightarrow c_{\beta,d}(\mu * p_t)(x), \quad (6)$$

uniformly for bounded  $\|\mu\|$ , bounded  $t^{-1}$ , and  $x \in \mathbb{R}^d$ .

Use  $\mu - x$  to denote the measure  $\mu$  shifted by  $-x$ . If  $\mu$  is finite, then by the scaling of  $\eta$ , (4), and Lemma 4.2, we can get the following chain of relations, which proves the uniform convergence of (6):

$$\begin{aligned} &\varepsilon^{2/\beta-d}(\beta t)^{-1/\beta} P_\mu \{ \eta_t B_x^\varepsilon > 0 \} \\ &= \varepsilon^{2/\beta-d}(\beta t)^{-1/\beta} P_{\mu-x} \{ \eta_t B_0^\varepsilon > 0 \} \\ &= \varepsilon^{2/\beta-d}(\beta t)^{-1/\beta} \int P_y \{ \eta_t B_0^\varepsilon > 0 \} (\mu - x)(dy) \\ &= \varepsilon^{2/\beta-d}(\beta t)^{-1/\beta} \int P_{y/\varepsilon} \{ \eta_{t/\varepsilon^2} B_0^1 > 0 \} (\mu - x)(dy) \\ &= \varepsilon^{-d} \int v_\infty(\varepsilon^{-2}t, \varepsilon^{-1}y) (\mu - x)(dy) \rightarrow c_{\beta,d}(\mu * p_t)(x). \end{aligned}$$

Let  $\mu$  be an infinite  $\sigma$ -finite measure satisfying  $\mu p_t < \infty$  for all  $t$ . From the proof of Lemma 2.5, we know that  $(\mu * p_{2t})(x) < \infty$  for any  $x \in \mathbb{R}^d$ . Then by dominated convergence based on Lemma 3.2(i), we can still get (6).

Now we turn to  $\xi_t$ . First note that by Lemma 3.1, as  $\varepsilon \rightarrow 0$ ,

$$\varepsilon^{2/\beta-d} P_\mu\{\xi_t B > 0\} \rightarrow c \Leftrightarrow \varepsilon^{2/\beta-d} (\beta t)^{-1/\beta} P_\mu\{\eta_t B > 0\} \rightarrow c, \quad (7)$$

$$\varepsilon^{2/\beta-d} P_\mu\{\xi_t^K B > 0\} \rightarrow c \Leftrightarrow \varepsilon^{2/\beta-d} a_t^{-1} P_\mu\{\eta_t^K B > 0\} \rightarrow c. \quad (8)$$

It remains to prove the uniform convergence for  $\xi_t$ . Since  $(\mu * p_t)(x) \leq t^{-d/2} \|\mu\|$ , we know that by (6),  $(\beta t)^{-1/\beta} P_\mu\{\eta_t B_x^\varepsilon > 0\} \rightarrow 0$ , uniformly for bounded  $\|\mu\|$ , bounded  $t^{-1}$ , and  $x \in \mathbb{R}^d$ . Then we may use Lemma 3.1 to get the uniform convergence for  $\xi_t$ .  $\square$

The following result, especially part (ii), will play a crucial role in Section 5. Here we approximate the hitting probabilities  $p_h^{K\varepsilon}$  by suitably normalized Dirac functions. This will be used in Lemma 5.1 to prove the Lebesgue approximation of  $\xi^K$ .

**Lemma 4.4** *Let  $p_h^\varepsilon(x) = P_x\{\eta_h B_0^\varepsilon > 0\}$ , where the  $\eta_h$  are clusters of a  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ . Recall that  $p_h^{K\varepsilon}(x) = P_x\{\eta_h^K B_0^\varepsilon > 0\}$ , where the  $\eta_h^K$  are clusters of  $\xi^K$ , the truncated  $K$ -process of  $\xi$ . Fix any bounded, uniformly continuous function  $f \geq 0$  on  $\mathbb{R}^d$ .*

(i) *As  $0 < \varepsilon^2 \ll h \rightarrow 0$ ,*

$$\left\| \varepsilon^{2/\beta-d} (\beta h)^{-1/\beta} (p_h^\varepsilon * f) - c_d f \right\| \rightarrow 0.$$

(ii) *Fix any  $b \in (0, 1/2)$ . Then as  $0 < \varepsilon^2 \ll h \rightarrow 0$  with  $\varepsilon^{2/\beta-d} h^{1+bd} \rightarrow 0$ ,*

$$\left\| \varepsilon^{2/\beta-d} a_h^{-1} (p_h^{K\varepsilon} * f) - c_d f \right\| \rightarrow 0.$$

*Both results hold uniformly over any class of uniformly bounded and equicontinuous functions  $f \geq 0$  on  $\mathbb{R}^d$ .*

*Proof:* (i) We follow the proof of Lemma 5.2(i) in [11]. By scaling of  $\eta$  and (6),

$$\varepsilon^{2/\beta-d} (\beta h)^{-1/\beta} \lambda^d p_h^\varepsilon = (\varepsilon/\sqrt{h})^{2/\beta-d} (\beta)^{-1/\beta} \lambda^d p_1^{\varepsilon/\sqrt{h}} \rightarrow c_{\beta,d}. \quad (9)$$

Defining  $\hat{p}_h^\varepsilon = p_h^\varepsilon / \lambda^d p_h^\varepsilon$ , we need to show that  $\|\hat{p}_h^\varepsilon * f - f\| \rightarrow 0$ . Writing  $w_f$  for the modulus of continuity of  $f$ , we get

$$\begin{aligned} \|\hat{p}_h^\varepsilon * f - f\| &= \sup_x \left| \int \hat{p}_h^\varepsilon(u) (f(x-u) - f(x)) du \right| \\ &\leq \int \hat{p}_h^\varepsilon(u) w_f(|u|) du \\ &\leq w_f(r) + 2\|f\| \int_{|u|>r} \hat{p}_h^\varepsilon(u) du. \end{aligned}$$

It remains to show that  $\int_{|u|>r} \hat{p}_h^\varepsilon(u) du \rightarrow 0$  for any fixed  $r > 0$ . Then notice that for any fixed  $r > 0$  by Lemma 3.2(i),

$$\varepsilon^{2/\beta-d}(\beta h)^{-1/\beta} \int_{|u|>r} p_h^\varepsilon(u) du \leq \int_{|u|>r} p_{2h}(u) du \rightarrow 0.$$

(ii) For  $p_h^{K\varepsilon}$ , Lemma 3.2(ii) yields for any fixed  $r > 0$ ,

$$\varepsilon^{2/\beta-d} a_h^{-1} \int_{|u|>r} p_h^{K\varepsilon}(u) du \leq \int_{|u|>r} p_{2h}(u) du \rightarrow 0.$$

Following the steps of the previous proof, it is enough to show that

$$\varepsilon^{2/\beta-d} a_h^{-1} \lambda^d p_h^{K\varepsilon} \rightarrow c_d. \quad (10)$$

Since  $\int_{|u|>h^b} p_{2h}(u) du \rightarrow 0$ , Lemma 3.2 yields

$$\varepsilon^{2/\beta-d}(\beta h)^{-1/\beta} 1\{(B_0^{h^b})^c\} \lambda^d p_h^\varepsilon \rightarrow 0, \quad \varepsilon^{2/\beta-d} a_h^{-1} 1\{(B_0^{h^b})^c\} \lambda^d p_h^{K\varepsilon} \rightarrow 0.$$

By (9), to prove (10) it suffices to show that

$$\varepsilon^{2/\beta-d}(\beta h)^{-1/\beta} 1\{B_0^{h^b}\} \lambda^d p_h^\varepsilon - \varepsilon^{2/\beta-d} a_h^{-1} 1\{B_0^{h^b}\} \lambda^d p_h^{K\varepsilon} \rightarrow 0,$$

or equivalently (by (7) and (8)),

$$\varepsilon^{2/\beta-d} \left( P_{1\{B_0^{h^b}\}\lambda^d} \{\xi_h B_0^\varepsilon > 0\} - P_{1\{B_0^{h^b}\}\lambda^d} \{\xi_h^K B_0^\varepsilon > 0\} \right) \rightarrow 0.$$

By Theorem 25.22 of [10] and (1),

$$\begin{aligned} & \varepsilon^{2/\beta-d} \left( P_{1\{B_0^{h^b}\}\lambda^d} \{\xi_h B_0^\varepsilon > 0\} - P_{1\{B_0^{h^b}\}\lambda^d} \{\xi_h^K B_0^\varepsilon > 0\} \right) \\ & \leq \varepsilon^{2/\beta-d} E_{1\{B_0^{h^b}\}\lambda^d} N_\xi([0, h], (K, \infty), \mathbb{R}^d) \\ & = \varepsilon^{2/\beta-d} E_{1\{B_0^{h^b}\}\lambda^d} \hat{N}_\xi([0, h], (K, \infty), \mathbb{R}^d) \\ & \stackrel{=}{=} \varepsilon^{2/\beta-d} E \int_0^h \|\xi_s\| ds \stackrel{=}{=} \varepsilon^{2/\beta-d} h^{1+bd} \rightarrow 0. \end{aligned}$$

□

## 5 Lebesgue approximations

To prove the Lebesgue approximation for a  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ , we begin with the Lebesgue approximation for  $\xi^K$ , the truncated  $K$ -process of  $\xi$ . Since  $\xi$  and  $\xi^K$  agree asymptotically as  $K \rightarrow \infty$ , we have thus proved the Lebesgue approximation for  $\xi$ . Write  $\tilde{c}_{\beta,d} = 1/c_{\beta,d}$  for convenience, where  $c_{\beta,d}$  is such as in Lemma 4.2. Recall that  $\xi_t^{K\varepsilon} = (\xi_t^K)^\varepsilon$ , the  $\varepsilon$ -neighborhood measure of  $\xi_t^K$ .

**Lemma 5.1** *Let  $\xi^K$  be the truncated  $K$ -process of a  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ . Fix any  $\mu \in \mathcal{M}_d$  and  $t > 0$ . Then under  $P_\mu$ , we have as  $\varepsilon \rightarrow 0$ :*

$$\tilde{c}_{\beta,d} \varepsilon^{2/\beta-d} \xi_t^{K\varepsilon} \xrightarrow{w} \xi_t^K \text{ a.s.}$$

*Proof:* We follow the proof of Theorem 7.1 in [11]. Fix any  $f \in C_K^d$ . Write  $\eta_h^{Ki}$  for the subclusters of  $\xi_t^K$  of age  $h$ . Since the ancestors of  $\xi_t^K$  at time  $s = t - h$  form a Cox process directed by  $\xi_s^K/a_h$ , Lemma 3.6(i) yields

$$E_\mu \left[ \sum_i \eta_h^{Ki\varepsilon} f \mid \xi_s^K \right] = a_h^{-1} \xi_s^K (p_h^{K\varepsilon} * f),$$

and so by Lemma 3.6(ii)

$$\begin{aligned} E_\mu \left| \sum_i \eta_h^{Ki\varepsilon} f - a_h^{-1} \xi_s^K (p_h^{K\varepsilon} * f) \right|^2 &= E_\mu \text{Var} \left[ \sum_i \eta_h^{Ki\varepsilon} f \mid \xi_s^K \right] \\ &\leq a_h \varepsilon^{d-2/\beta} h^{d/2} \|f\|^2 E_\mu \|\xi_s^K / a_h\| \\ &\leq \varepsilon^{d-2/\beta} h^{d/2} \|f\|^2 \|\mu\|, \end{aligned}$$

where the last inequality follows from  $E_\mu \|\xi_s^K\| \leq \|\mu\|$ . Combining with Lemma 3.7(i) gives

$$\begin{aligned} E_\mu \left| \xi_t^{K\varepsilon} f - a_h^{-1} \xi_s^K (p_h^{K\varepsilon} * f) \right| &\leq E_\mu \left| \xi_t^{K\varepsilon} f - \sum_i \eta_h^{Ki\varepsilon} f \right| + E_\mu \left| \sum_i \eta_h^{Ki\varepsilon} f - a_h^{-1} \xi_s^K (p_h^{K\varepsilon} * f) \right| \\ &\leq \varepsilon^{2(d-2/\beta)} h^{1-d/2} \|f\| + \varepsilon^{1/2(d-2/\beta)} h^{d/4} \|f\| \\ &= \varepsilon^{d-2/\beta} \left( \varepsilon^{d-2/\beta} h^{1-d/2} + \varepsilon^{-1/2(d-2/\beta)} h^{d/4} \right) \|f\|. \end{aligned}$$

Let  $c$  satisfy

$$(d - 2/\beta) + (-d/2 + 1/2)c = 0. \quad (11)$$

Clearly  $c \in (0, 2)$ . Taking  $\varepsilon = r^n$  for a fixed  $r \in (0, 1)$  and  $h = \varepsilon^c = r^{cn}$ , and writing  $s_n = t - h = t - r^{cn}$ , we obtain

$$\begin{aligned} E_\mu \sum_n r^{n(2/\beta-d)} \left| \xi_t^{Kr^n} f - a_{r^{cn}}^{-1} \xi_{s_n}^K (p_{r^{cn}}^{Kr^n} * f) \right| &\leq \sum_n \left( r^{[(d-2/\beta)+(-d/2+1)c]n} + r^{[-1/2(d-2/\beta)+(d/4)c]n} \right) \|f\| < \infty, \end{aligned}$$

since  $(d - 2/\beta) + (-d/2 + 1)c > 0$  and  $-1/2(d - 2/\beta) + (d/4)c > 0$  by (11). This implies

$$r^{n(2/\beta-d)} \left| \xi_t^{Kr^n} f - a_{r^{cn}}^{-1} \xi_{s_n}^K (p_{r^{cn}}^{Kr^n} * f) \right| \rightarrow 0 \text{ a.s. } P_\mu. \quad (12)$$

Now we write

$$\begin{aligned} &\left| \varepsilon^{2/\beta-d} \xi_t^{K\varepsilon} f - c_{\beta,d} \xi_t^K f \right| \\ &\leq \varepsilon^{2/\beta-d} \left| \xi_t^{K\varepsilon} f - a_h^{-1} \xi_s^K (p_h^{K\varepsilon} * f) \right| + c_{\beta,d} |\xi_s^K f - \xi_t^K f| \\ &\quad + \|\xi_s^K\| \left\| \varepsilon^{2/\beta-d} a_h^{-1} (p_h^{K\varepsilon} * f) - c_{\beta,d} f \right\|. \end{aligned}$$



For the last term, we first fix  $b = 1/2 - 1/d$ , then apply Lemma 4.4. Noting that by (11)

$$(2/\beta - d) + (1 + bd)c = (2/\beta - d) + (d/2)c > 0,$$

we get by Lemma 4.4

$$\left\| \varepsilon^{2/\beta-d} a_h^{-1} (p_h^{K\varepsilon} * f) - c_{\beta,d} f \right\| \rightarrow 0$$

along the sequence  $(r^n)$ . Using (12) and the a.s. weak continuity of  $\xi^K$  at the fixed time  $t$ , we see that the right-hand side tends a.s. to 0 as  $n \rightarrow \infty$ , which implies  $\varepsilon^{2/\beta-d} \xi_t^{K\varepsilon} f \rightarrow c_{\beta,d} \xi_t^K f$  a.s. as  $\varepsilon \rightarrow 0$  along the sequence  $(r^n)$  for any fixed  $r \in (0, 1)$ . Since this holds simultaneously, outside a fixed null set, for all rational  $r \in (0, 1)$ , the a.s. convergence extends by Lemma 2.3 in [11] to the entire interval  $(0, 1)$ .

Applying this result to a countable, convergence-determining class of functions  $f$  (cf. Lemma 3.2.1 in [2]), we obtain the required a.s. vague convergence. Since  $\mu$  is finite, the  $(2, \beta)$ -process  $\xi_t$  has a.s. compact support (cf. Theorem 9.3.2.2 of [2] and the proof of Theorem 1.2 in [3]). By Lemma 2.2,  $\xi_t^K$  also has a.s. compact support, and so the a.s. convergence remains valid in the weak sense.  $\square$

Now we may prove our main result, the Lebesgue approximation of  $(2, \beta)$ -processes. Again, we write  $\tilde{c}_{\beta,d} = 1/c_{\beta,d}$  for convenience, where  $c_{\beta,d}$  is such as in Lemma 4.2. Also recall that  $\xi_t^\varepsilon = (\xi_t)^\varepsilon$  denotes the  $\varepsilon$ -neighborhood measure of  $\xi_t$ . For random measures  $\xi_n$  and  $\xi$  on  $\mathbb{R}^d$ ,  $\xi_n \xrightarrow{v} \xi$  (or  $\xrightarrow{w}$ ) in  $L^1$  means that  $\xi_n f \rightarrow \xi f$  in  $L^1$  for all  $f$  in  $C_K^d$  (or  $C_b^d$ ).

**Theorem 5.2** *Let the  $(2, \beta)$ -process  $\xi$  in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$  be locally finite under  $P_\mu$ , and fix any  $t > 0$ . Then under  $P_\mu$ , we have as  $\varepsilon \rightarrow 0$ :*

$$\tilde{c}_{\beta,d} \varepsilon^{2/\beta-d} \xi_t^\varepsilon \xrightarrow{v} \xi_t \text{ a.s and in } L^1.$$

*This remains true in the weak sense when  $\mu$  is finite. The weak version holds even for the clusters  $\eta_t$  when  $\|\mu\| = 1$ .*

*Proof:* For a finite initial measure  $\mu$ , by Lemma 5.1 and Lemma 2.1 we get as  $\varepsilon \rightarrow 0$

$$\tilde{c}_{\beta,d} \varepsilon^{2/\beta-d} \xi_t^\varepsilon \xrightarrow{w} \xi_t \text{ a.s.}$$

For a general  $\mu \in \mathcal{M}_d$  with  $\mu p_t < \infty$  for all  $t > 0$ , write  $\mu = \mu' + \mu''$  for a finite  $\mu'$ , and let  $\xi = \xi' + \xi''$  be the corresponding decomposition of  $\xi$  into independent components with initial measures  $\mu'$  and  $\mu''$ . Fixing an  $r > 1$  with  $\text{supp } f \subset B_0^{r-1}$  and using the result for finite  $\mu$ , we get a.s. on  $\{\xi_t'' B_0^r = 0\}$

$$\varepsilon^{2/\beta-d} \xi_t^\varepsilon f = \varepsilon^{2/\beta-d} \xi_t'^\varepsilon f \rightarrow c_{\beta,d} \xi_t' f = c_{\beta,d} \xi_t f.$$

As  $\mu' \uparrow \mu$ , we get by Lemma 3.4

$$P_\mu\{\xi_t'' B_0^r = 0\} = P_{\mu''}\{\xi_t B_0^r = 0\} \rightarrow 1,$$

and the a.s. convergence extends to  $\mu$ . As in the proof of Lemma 5.1, we can obtain the required a.s. vague convergence.

To prove the convergence in  $L^1$ , we note that for any  $f \in C_K^d$

$$\begin{aligned} \varepsilon^{2/\beta-d} E_\mu \xi_t^\varepsilon f &= \varepsilon^{2/\beta-d} \int P_\mu\{\xi_t B_x^\varepsilon > 0\} f(x) dx \\ &\rightarrow \int c_{\beta,d}(\mu * p_t)(x) f(x) dx = c_{\beta,d} E_\mu \xi_t f, \end{aligned} \quad (13)$$

by Theorem 4.3. Combining this with the a.s. convergence under  $P_\mu$  and using Proposition 4.12 in [10], we obtain  $E_\mu |\varepsilon^{2/\beta-d} \xi_t^\varepsilon f - c_{\beta,d} \xi_t f| \rightarrow 0$ . For finite  $\mu$ , (13) extends to any  $f \in C_b^d$  by dominated convergence based on Lemmas 3.1 and 3.2(i), together with the fact that  $\lambda^d(\mu * p_t) = \|\mu\| < \infty$  by Fubini's theorem.

To extend the Lebesgue approximation to the individual clusters  $\eta_t$ , let  $\zeta_0$  denote the process of ancestors of  $\xi_t$  at time 0, and note that

$$P_x\{\eta_t \in \cdot\} = P_{\delta_x}[\xi_t \in \cdot | \|\zeta_0\| = 1],$$

where  $P_{\delta_x}\{\|\zeta_0\| = 1\} = (\beta t)^{-1/\beta} e^{-(\beta t)^{-1/\beta}} > 0$ . The a.s. convergence then follows from the corresponding statement for  $\xi_t$ . Since

$$P_\mu\{\eta_t \in \cdot\} = \int \mu(dx) P_x\{\eta_t \in \cdot\},$$

the a.s. convergence under any  $P_\mu$  with  $\|\mu\| = 1$  also follows. To obtain the weak  $L^1$ -convergence in this case, we note that for  $f \in C_b^d$ ,

$$\begin{aligned} \varepsilon^{2/\beta-d} E_\mu \eta_t^\varepsilon f &= \varepsilon^{2/\beta-d} \int P_\mu\{\eta_t B_x^\varepsilon > 0\} f(x) dx \\ &\rightarrow c_{\beta,d}(\beta t)^{1/\beta} \int (\mu * p_t)(x) f(x) dx = c_{\beta,d} E_\mu \eta_t f, \end{aligned}$$

by dominated convergence based on Lemma 3.2(i) and Theorem 4.3.  $\square$

As in Corollary 7.2 of [11], for the intensity measures in Theorem 5.2, we have even convergence in total variation.

**Corollary 5.3** *Let  $\xi$  be a  $(2, \beta)$ -process in  $\mathbb{R}^d$  with  $\beta < 1$  and  $d > 2/\beta$ . Then for any finite  $\mu$  and  $t > 0$ , we have as  $\varepsilon \rightarrow 0$ :*

$$\left\| \varepsilon^{2/\beta-d} E_\mu \xi_t^\varepsilon - c_{\beta,d} E_\mu \xi_t \right\| \rightarrow 0.$$

*This remains true for the clusters  $\eta_t$ , and it also holds locally for  $\xi_t$  whenever  $\xi$  is locally finite under  $P_\mu$ .*

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